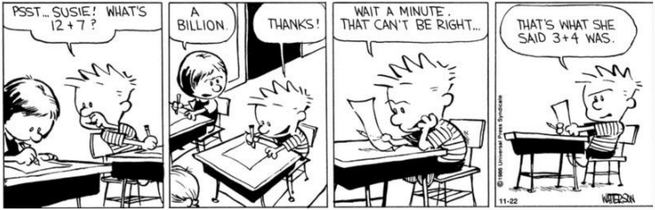
# Games, graphs, and machines

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# Recall the definitions

- $[0]$ = The set $\{0\}$.
- $[1]$ = The set of all positive numbers.

$[0] + [0] =$ [0]

$[0] + [1] =$ [1]

$[1] + [1] =$ [1]

$=$ [15]

$[0] \cdot [0] =$ [0]

$[0] \cdot [1] =$ [0]

$[1] \cdot [1] =$ [1]

[0] = False

[1] = True

+ : "or"

$\cdot$ : "and"

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Find $A^{*k}$ for $k = 1, 2, 3, \ldots$.

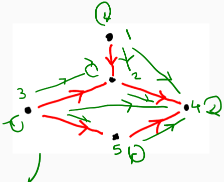

Can you explain the pattern using a graph?

$A^{*2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad A^{*3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad A^{*4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \ldots$

<u>Thm</u>: $A^{*k}_{ij} = \begin{cases} 1 & \text{if there is a path from } i \text{ to } j \text{ of length } k \text{ (} k \text{ edges)} \\ 0 & \text{otherwise} \end{cases}$

# Existence of paths

Let $A$ be the adjacency matrix of the graph.




- Find $A^{*k}$ for large $k$.
- Find $\underbrace{I + A + A^{*2} + \cdots + A^{*k}}_{\text{Detects paths of length up to K.}}$ for large $k$.

Has no directed cycles.

$\Rightarrow$ No path with 6 vertices (5 edges)

$\Rightarrow$ $A^{*5} = 0$ $A^{*6} = 0$ . . . .

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 | 0 |
| 2 | 0 | 1 | 0 | 1 | 0 |
| 3 | | | 1 | | |
| 4 | | | | 1 | |
| 5 | | | | | 1 |

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Obs: $A + A^{*2} + A^{*3} + \ldots\ldots$

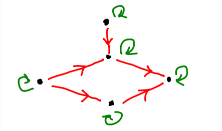
↳ the diagonal entries detect directed cycles.

# Adding loops

Suppose we add loops.

Detects length k paths.

$\parallel$



- Now what is $A^{*k}$ for large $k$?
- What about $\underbrace{I + A + A^{*2} + \cdots + A^{*k}}_{= A^{*k}}$? (loops)

$A = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$ $\quad A^3 = \begin{pmatrix} 1 & 3 \\ & 1 \end{pmatrix} \; \ldots$

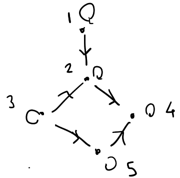
$A^{*2} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ $\quad A^{*3} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} = \ldots$

loops $\Rightarrow$ Existence of length k path
$\parallel$
Existence of length $\leq$ k path.

$I + A + A^2 = \begin{pmatrix} 3 & 3 \\ & 3 \end{pmatrix}$

$A^2 = \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ in Boolean Arith




$A^{*K}$ large K

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & & \vdots & & & \\ 3 & & \vdots & & & \\ 4 & & \vdots & & & \\ 5 & & & & & \end{array}$$

Large K

?

All $K \geq 2$

$K \geq n-1$.

For any G, $1 + A + A^{*2} + A^{*3} + \dots + A^{*K}$ will stabilise.

n vertices

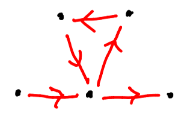
entries detect the existence of paths.

If you can reach from i to j at all, then you can reach by taking path of length $\leq n-1$ ($\leq n$ vertices).

$\Rightarrow$ adding $A^{*n}$, $A^{*(n+1)}$ above is superfluous!

# Modular arithmetic?

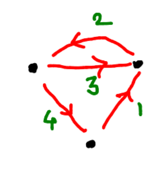
Let $A$ be the adjacency matrix of



Can you describe $A^{*k}$ for all $k$?

# Min plus

Write the min/plus weighted adjacency matrix of




Assume that the loops have weight 0 (not shown).

# Min/plus arithmetic

Find

$$3 \odot (4 \oplus 1) \oplus 1 \odot (\infty \oplus 3).$$